

Targeted Learning with Application in Rare Disease Trials

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8th New England Rare Disease Statistics (NERDS),
October 10, 2025

Acknowledgements: Rachael Phillips, Tianyue Zhou, Susan Gruber, Ivana Malenica, Sky Qiu, Lei Nie,

Wonyul Lee, Hana Lee

Traditional toolbox for statistics: Recipe oriented, enforces false constraints, not made for Big Data

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Goal	Type of Data			
	Measurement (from Gaussian Population)	Rank, Score, or Measurement (from Non-Gaussian Population)	Binomial (Two Possible Outcomes)	Survival Time
Describe one group	Mean, SD	Median, interquartile range	Proportion	Kaplan Meier survival curve
Compare one group to a hypothetical value	One-sample t test	Wilcoxon test	Chi-square or Binomial test**	
Compare two unpaired groups	Unpaired t test	Mann-Whitney test	Fisher's test (chi-square for large samples)	Log-rank test or Mantel-Haenszel*
Compare two paired groups	Paired t test	Wilcoxon test	McNemar's test	Conditional proportional hazards regression*
Compare three or more unmatched groups	One-way ANOVA	Kruskal-Wallis test	Chi-square test	Cox proportional hazard regression**
Compare three or more matched groups	Repeated-measures ANOVA	Friedman test	Cochrane Q**	Conditional proportional hazards regression**
Quantify association between two variables	Pearson correlation	Spearman correlation	Contingency coefficients**	
Predict value from another measured variable	Simple linear regression or Nonlinear regression	Nonparametric regression**	Simple logistic regression*	Cox proportional hazard regression*
Predict value from several measured or binomial variables	Multiple linear regression* or Multiple nonlinear regression**		Multiple logistic regression*	Cox proportional hazard regression*

Performance of traditional tools: Coverage of Confidence Intervals deteriorates with sample size

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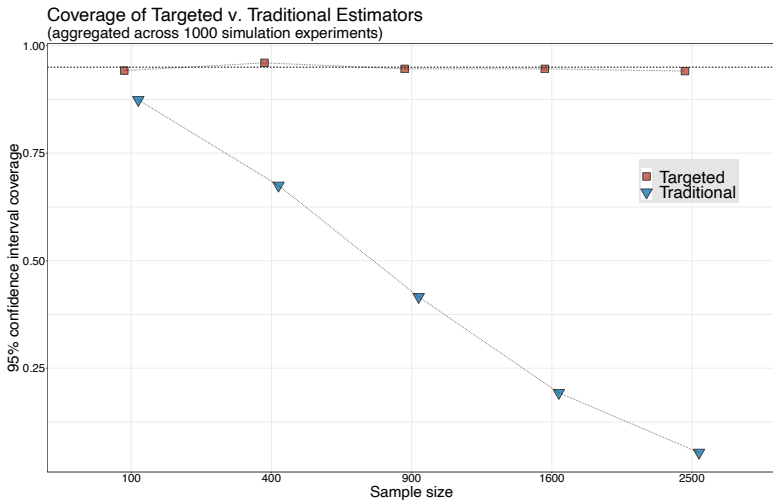
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Performance of traditional tools: Type I error deteriorates with sample size

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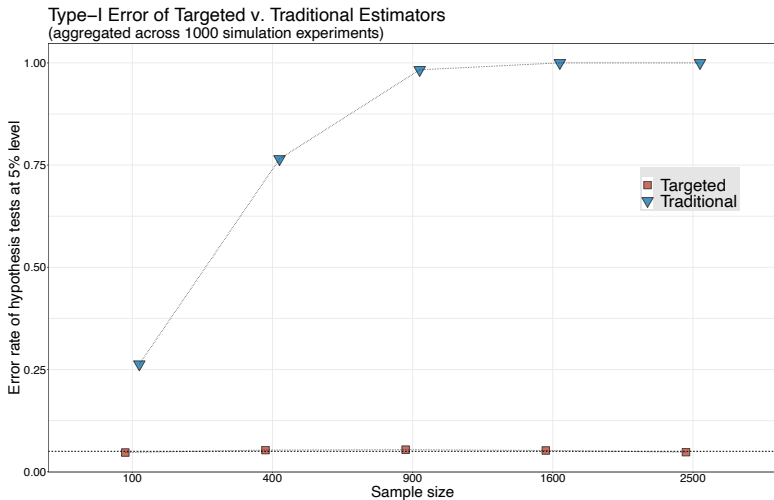
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Traditional tools invite/encourage post-hoc model manipulation

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Why care about statistical inference?

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Why Most Published Research Findings Are False

John P. A. Ioannidis

**False-Positive Psychology: Undisclosed
Flexibility in Data Collection and Analysis
Allows Presenting Anything as Significant**

Joseph P. Simmons¹, Leif D. Nelson², and Uri Simonsohn¹

¹The Wharton School, University of Pennsylvania, and ²Haas School of Business, University of California, Berkeley

The Statistical Crisis in Science

Data-dependent analysis—a “garden of forking paths”—explains why many statistically significant comparisons don’t hold up.

Andrew Gelman and Eric Loken

Targeted Learning for answering statistical and causal questions with confidence intervals

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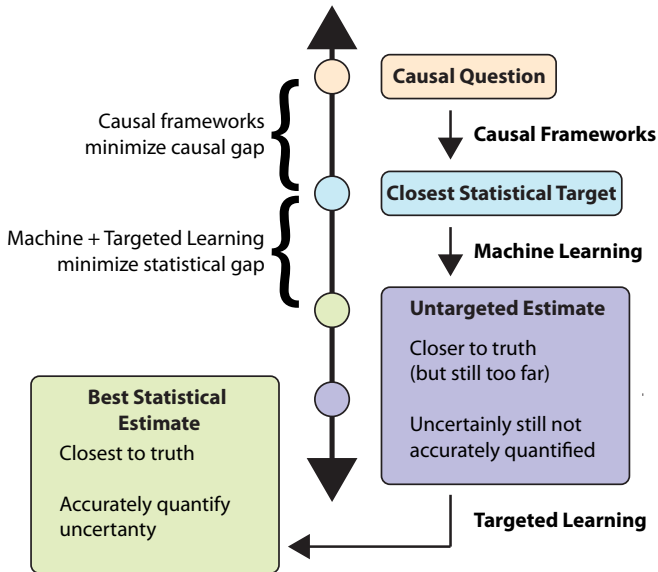
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Targeted Learning is a subfield of statistics

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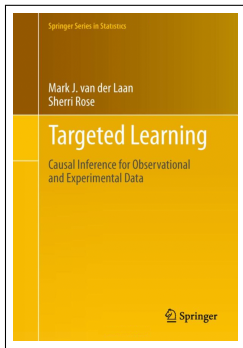
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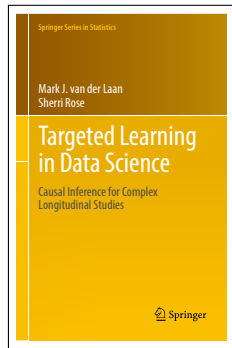
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van der Laan & Rose, *Targeted Learning: Causal Inference for Observational and Experimental Data*. New York: Springer, 2011.



van der Laan & Rose, *Targeted Learning in Data Science: Causal Inference for Complex Longitudinal Studies*. New York: Springer, 2018.

The Hitchhiker's Guide to the tLverse

Better clinical decisions from observational data

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Statistics
in Medicine

Research Article

Received 24 May 2013,

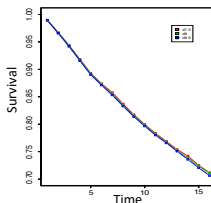
Accepted 5 January 2014

Published online 17 February 2014 in Wiley Online Library

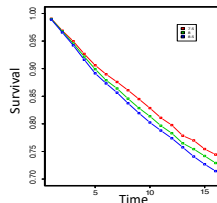
(wileyonlinelibrary.com) DOI: 10.1002/sim.6099

Targeted learning in real-world comparative effectiveness research with time-varying interventions

Romain Neugebauer,^{a,*†} Julie A. Schmittdiel^a and Mark J. van der Laan^b



Standard methods: No benefit to more aggressive intensification strategy



Targeted Learning: More aggressive intensification protocols result in better outcomes



FDA Sentinel Innovation Center

- **Safety evaluation with high dimensional data:**

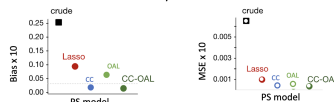
Wyss et al. (2024), AJE, Targeted Learning with an Undersmoothed Lasso Propensity Score Model for Large Scale Covariate Adjustment in Healthcare Database Studies.

- **Subset calibration/two-stage designs:**

-Ongoing project evaluating methods such as the two-stage design TMLE for study designs that involve a subset of subjects with carefully curated confounders and or outcomes, and a remaining set of subjects.

-This is a common type of design to obtain desired causal identification from RWD while still gaining efficiency from the less curated data set.

Plasmode study results



Collaborative control greatly reduced bias and improved MSE

- Less regularization captured more relevant confounder information in PS

These projects involve multi-author working groups with FDA/Pharma/Academics/Kaiser Permanente.

The Sentinel Innovation Center is funded by the FDA through the Department of Health and Human Services (HHS) Task order 75F40119D10037.

Comparative Effectiveness: Targeted Learning FDA Demonstration Project

Resulted in various collaborative relations with FDA statisticians

TL for RWE

multi-year FDA-funded project

TL on YouTube



Publications with FDA co-authors

Targeted learning: Towards a future informed by real-world evidence

Suzan Gruber, Rachael Y. Phillips, Hana Lee, Martin Ho, John Concato, Mark J. van der Laan

The 21st Century Cures Act of 2016 includes a provision for the U.S. Food and Drug Administration (FDA) to evaluate the potential use of real-world evidence (RWE) to support new indications for all “covered products”.

Evaluating and improving real-world evidence with Targeted Learning

Suzan Gruber, Rachael Y. Phillips, Hana Lee, John Concato, Mark van der Laan

Reason: The Targeted Learning method provides a common guide for generating and evaluating real-world evidence (RWE) from observational data. While observational data can be useful in certain situations, they have a number of limitations: observational data are often noisy, observational data are often incomplete, and observational data are often biased. The Targeted Learning method provides a common guide for generating and evaluating real-world evidence (RWE) from observational data.

Developing a Targeted Learning-Based Statistical Analysis Plan

Suzan Gruber, Rachael Y. Phillips, Hana Lee, John Concato, Mark van der Laan

Over 250 FDA short course attendees

SHORT COURSE ANNOUNCEMENT



A Targeted Learning Framework for Causal Effect Estimation using Real-World Data

Funded by FDABAA-19-00123-A3



Statistical challenges with RWD

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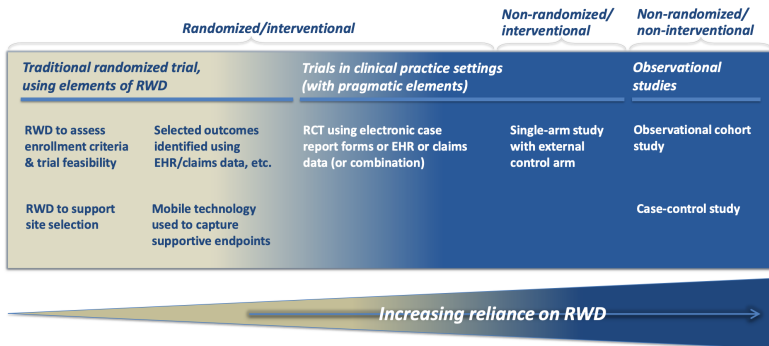
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Randomized/interventional		Non-randomized/ interventional	Non-randomized/ non-interventional	
Traditional randomized trial, using elements of RWD		Trials in clinical practice settings (with pragmatic elements)	Observational studies	
RWD to assess enrollment criteria & trial feasibility	Selected outcomes identified using EHR/claims data, etc.	RCT using electronic case report forms or EHR or claims data (or combination)	Single-arm study with external control arm	Observational cohort study
RWD to support site selection	Mobile technology used to capture supportive endpoints			Case-control study

RWD Challenges

- ☐ Selection bias
- ☐ Intercurrent events
- ☐ Informative missingness
- ☐ Treatment by indication
- ☐ High dimensional covariates
- ☐ Outcome measurement error
- ☐ Statistical model misspecification
- ☐ Differences between external
controls and single trial arm RCT

*Targeted Learning
path supports regulatory
decision making*

The roadmap for targeted learning from data

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STEP 1:
DESCRIBE
EXPERIMENT

STEP 2:
SPECIFY STATISTICAL
MODEL

STEP 3:
DEFINE STATISTICAL
QUERY

STEP 4:
CONSTRUCT
ESTIMATOR

STEP 5:
OBTAIN
INFERENCE

STEP 6:
MAKE SUBSTANTIVE
CONCLUSION

Targeted Maximum Likelihood Estimation (TMLE)

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TMLE

1

Initial estimation of $E[Y|A, W]$
with super (machine) learning

2

Updating initial estimate to achieve
optimal bias-variance trade-off for ψ_{stat}

TMLE estimates are optimal:

plug-in, efficient, unbiased, finite sample robust

TMLE Step 1: Super learner

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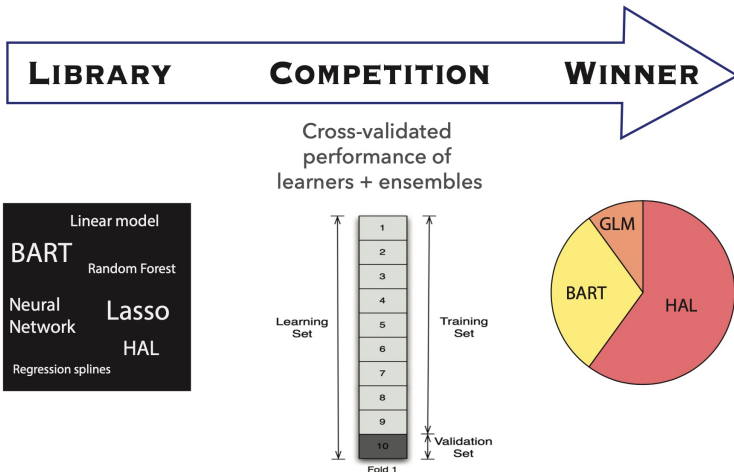
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Hugely advantageous when coupled with NLP-derived covariates with EHR

TMLE Step 2: Targeting follows a path of maximal change in target estimand per unit likelihood

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Targeted Learning with RWD

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- ☐ Statistical model misspecification
- ☐ Differences between external controls and single trial arm RCT

Targeted Learning path supports regulatory decision making

Targeted Learning

- ✓ Roadmap for causal and statistical inference
- ✓ Realistic statistical model
- ✓ Statistical estimand approximates answer to causal question
- ✓ Flexible estimation and dimension reduction with Super Learner
- ✓ Model-free sensitivity analysis
- ✓ Generate RWE with confidence

A typical rare disease RCT (FDA/TLRev/UCBerkeley collaboration)

- On each subject we observe baseline history W ; a binary randomized treatment A ; at multiple visit times an outcome process $Y(t)$ at $t = 1, \dots, \tau$.
- The outcome $Y(t)$ has multiple components, $Y_k(t)$, $k = 1, \dots, K$.
- Sample size small, e.g. $n = 50$, $g_0(1|W) = P_0(A = 1) = 2/3$.
- One defines **some** composite outcome such as a sum of scores $\bar{Y} = \sum_{k=1}^K Y(k)$ and define the causal estimand $\Psi(P_0)$ as the ATE on $\bar{Y}$.
- A TMLE involves super-learning fit $\bar{Q}_n$ of $E_0(Y | W, A)$; a targeted update $\bar{Q}_n^*$ involving true PS $g_0(1|W)$, and plug-in estimator $1/n \sum_i \{\bar{Q}_n^*(1, W_i) - \bar{Q}_n^*(0, W_i)\}$.
- Such a TMLE is unbiased (due to DR) and typically heavily outperforms a simple unadjusted estimator of the ATE.

Simulations imitating real RCT from Zevra

Simulation Setup:

- Sample size: $n = 50$.
- 5 baseline covariates: 2 continuous, 2 binary, 1 count.
- 2:1 randomization (treatment:control).
- Outcome: Sum of multi-domain assessment scores (0-20).
True outcome model is nonlinear with 2 prognostic covariates (often not known due to poor natural history understanding, necessitates the use of SL).

Results (True ATE = -1.486; 500 simulations):

Method	Bias	SE	MSE	Coverage	Power
Unadjusted	-0.096	1.829	3.356	0.946	0.118
ANCOVA	0.019	0.424	0.180	0.940	0.910
TMLE+SL	-0.016	0.369	0.137	0.940	0.988

TMLE+SL achieves efficiency gains over fixed adjustment methods (e.g., ANCOVA) while maintaining valid inference.

Right-censoring of outcome process in RCT

- Typically, a percentage of the subjects are right-censored.
- Forward imputation of outcome after drop-out is problematic.
- A common method is mixed linear repeated measures models (MLRM).
- MLRM remains consistent under informative right-censoring if the model for $E(Y(t)|W, A) = m_{\beta}(t, W, A)$ is correctly specified. However, in general, it is inconsistent due to informative drop-out.
- Instead, one might use `ltmle()` to evaluate the ATE in the world in which subjects are uncensored till end-point.
- Pros: Allows informative drop-out; utilizes SL to gain efficiency.

Simulations of ltmle() versus MLRM

Simulation Setup:

- Sample size, baseline covariates and treatment assignment same as in the point treatment setting.
- Longitudinal data with 4 time points
- Informative censoring depends on treatment and last observed outcome.
- True outcome process remains nonlinear (MLRM is misspecified).

Results (True ATE = -1.499; 500 simulations):

Method	Bias	SE	MSE	Coverage	Power
MLRM	-0.137	0.526	0.296	0.896	0.874
L-TMLE+SL	0.045	0.469	0.222	0.950	0.864

MLRM yields biased estimates and invalid inference due to mean model misspecification. L-TMLE remains unbiased and maintains nominal coverage (efficiency gain due to SL).

Multiple testing Challenge

- One is concerned about just defining some sum score outcome or other choice.
- We might lose power by that choice.
- One could define an ATE $\Psi_k(P_0)$ for each outcome $Y(k)$, $k = 1, \dots, K$.
- Compute TMLE
$$\psi_n^*(k) = 1/n \sum_i \{ \bar{Q}_{k,n}^*(1, W_i) - \bar{Q}_{k,n}^*(0, W_i) \}.$$
- A vector TMLE $(\psi_n^*(k) : k = 1, \dots, K)$ satisfies $n^{1/2}(\psi_n^* - \psi_0)/\sigma_n \Rightarrow_d N(0, \Sigma_0)$ with Σ_0 being correlation matrix of the vector influence curve $(D_{\Psi_{k()}, P_0}^* / \sigma(k) : k)$ of the TMLE.

Multivariate Normal Null Distribution from Influence Curve TMLE (Dudoit, vdL MT book)

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- $N(0, \Sigma_0)$ represents the right null distribution for multiple testing of $H_0(k) : \Psi_k(P_0) = 0$, $k = 1, \dots, K$. One sets the cut-offs of the t -statistics $n^{1/2}(\psi_n^*(k) - \psi_0(k))/\sigma_n(k)$, $k = 1, \dots, K$, so that the FWE or any other generalized type I error is controlled at level 0.05 under sampling the t -statistic vector from $N(0, \Sigma_0)$.
- Similarly, one can use this null distribution $N(0, \Sigma_0)$ in a step-down multiple testing procedure.
- Using quantile-quantile function one can transform the marginal distributions of $N(0, \rho_0)$ into marginal distributions controlled by user such as a permutation distribution.
- Even though this is much more powerful than Bonferroni, small sample sizes generally imply lack of power for any multiple testing procedure.

Using Max-t statistic to test overall null $\Psi(P_0) = 0$

- To avoid multiple testing, one aims to test the overall null $H_0 : \psi_0(k) = 0, k = 1, \dots, K$.
- In short: $H_0 : \psi_0 = 0$.
- One could do that with the max-t statistic applied to standardized TMLE and setting cut-off to control type-I error under the $N(0, \Sigma_0)$ -distribution of the t-statistic.
- Much better than Bonferoni, but still price to pay in power!

Define α -specific ATE and test of its null $H_0(\alpha)$

- Let $\bar{Y}(\alpha) \equiv \sum_{k=1}^K \alpha(k) Y(k)$.
- Define ATE

$$\psi_0(\alpha) = E_0\{E_0(\bar{Y}(\alpha) \mid A = 1, W) - E_0(\bar{Y}(\alpha) \mid A = 0, W)\}.$$

- The TMLE of $\psi_0(\alpha)$ using an inconsistent estimator $\bar{Q}_{\alpha,n}^* = \sum_{k=1}^K \alpha(k) \bar{Q}_{k,n}^*$ of $\bar{Q}_{0,\alpha} = E_0(\bar{Y}(\alpha) \mid A, W) = \sum_k \alpha(k) E_0(Y(k) \mid W, A)$ is asymptotically linear with influence curve $\sum_{k=1}^K \alpha(k) D_{\Psi_k(\cdot), \bar{Q}, g_0}^*$.
- $\bar{Q} = (\bar{Q}_k : k)$ represents limit of outcome regressions.
- The asymptotic variance of the standardized TMLE $\psi_n^*(\alpha)$:

$$\begin{aligned}\sigma_{\alpha,0}^2 &= \sigma_{\alpha}^2(\bar{Q}, g_0, P_0) = \alpha^\top \Sigma_0 \alpha \\ \Sigma_0(k_1, k_2) &= P_0 D_{\Psi_{k_1}(\cdot), \bar{Q}, g_0}^* D_{\Psi_{k_2}(\cdot), \bar{Q}, g_0}^*.\end{aligned}$$

Oracle test

- The power of the test $n^{1/2}\psi_n^*(\alpha)/\sigma_{\alpha,0} > 1.65$ at $\psi_0(\alpha)$ is given by

$$\bar{\Phi}(1.65 - n^{1/2}\psi_0(\alpha)/\sigma_{\alpha,0})$$

with $\bar{\Phi}(x) = P(N(0, 1) > x)$.

- Define

$$\alpha_0 = \arg \max_{\alpha} \psi_0(\alpha)/\sigma_{\alpha,0}.$$

- Define oracle shift $\theta_{\alpha,0} \equiv \psi_0(\alpha)/\sigma_{\alpha,0}$.
- Then $H_0(\alpha_0) : \psi_0(\alpha_0) = 0$ implies the most powerful test among all tests.
- We have

$$\alpha_0 = \Sigma_0^{-1}(\psi_0),$$

with $\psi_0 = (\psi_0(k) : k = 1, \dots, K)$.

Estimated oracle test statistic

- Suppose we estimate α_0 with $\alpha_n = \Sigma_n^{-1} \psi_n^*$.
- The t-statistic for $H_0(\alpha_n)$ is given by $t_n = n^{1/2} \alpha_n^\top \psi_n^* / \sigma_{\alpha_n, n}$ which can be written as

$$t_n = n^{1/2} \left(\psi_n^{*, \top} \Sigma_n^{-1} \psi_n^* \right)^{1/2}.$$

- Thus the estimate of the oracle t-statistic yields a TMLE-based Hotelling Chi-square statistic:

$$t_n^2 = n \left(\psi_n^{*, \top} \Sigma_n^{-1} \psi_n^* \right)^{1/2} \sim_{H_0} \chi_K^2.$$

- This suggests that latter test is highly powerful!

Data adaptive target parameter using sample splitting

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- Use V -fold sample splitting with empirical measures $P_{n,v}, P_{n,v}^1, v = 1, \dots, V$, for training and validation sample across splits.
- Compute $\hat{\alpha}(P_{n,v})$ of oracle α_0 based on training sample.
- Consider data adaptive target parameter

$$\theta_{n,v,0} = \alpha_{n,v}^\top \psi_0 / \sigma_{\alpha_{n,v}}(\bar{Q}, g_0, P_0).$$

- Due to $\frac{d}{d\alpha_0} \alpha_0^\top \psi_0 / \sigma_{\alpha,0} = 0$ we have that

$$\theta_{n,v,0} \approx \alpha_0^\top \psi_0 / \sigma_{\alpha_0,0}$$

in first order!

Cross-fitted TMLE of data adaptive target parameter

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- Compute a cross-fitted TMLE $\theta_{n,v,n}^* \equiv \alpha_{n,v}^\top \psi_{n,v}^* / \sigma_{\alpha_{n,v},n}$ of this $\theta_{n,v,0}$.
- Here the TMLE $\psi_{n,v}^*$ gets its initial estimator still from training sample but targeting is carried out on validation sample $P_{n,v}^1$.
- The latter targeting step can be pooled across sample splits v .
- One can estimate $\sigma_{\alpha_{n,v},0}$ based on whole sample, no sample splitting needed.
- The CV-TMLE of $\theta_{\hat{\alpha},0} = 1/V \sum_v \theta_{n,v,0}$ is defined as the average:

$$\theta_{\hat{\alpha},cv-tmle,n} = \frac{1}{V} \sum_{v=1}^V \theta_{n,v,n}^*$$

Asymptotics of CV-TMLE, Inference for fixed oracle parameter as well

- **Suppose that the overall null does not hold.**
- Then α_n converges to the oracle choice α_0 .
- One can prove (without Donsker class condition) that

$$\theta_{\hat{\alpha}, cv-tmle, n} - 1/V \sum_{v=1}^V \theta_{\alpha_{n,v}, 0} = P_n D_{\hat{\theta}_{\alpha_0(\cdot), P_0}}^* + o_P(n^{-1/2}),$$

for influence curve of $\hat{\theta}_{\alpha_0}(P_n)$ treating α_0 as fixed/known.

- Thus, it provides inference for $1/V \sum_{v=1}^V \alpha_{n,v}^\top \psi_0 / \sigma_{\alpha_{n,v}, 0}$ (confidence interval and test).
- Moreover, under $o_P(n^{-1/4})$ -consistency of $\hat{\alpha}(P_n)$ (we have $n^{-1/2}$) yields that it also yields inference for the fixed parameter $\alpha_0^\top \psi_0 / \sigma_{\alpha_0, 0}$.
- It provides a test of $H_0 : \alpha_0^\top \psi_0 = 0$ and thus $H_0 : \psi_0(k) = 0$ for all $k = 1, \dots, K$, as long as the overall null H_0 is not true.
- Simulations (Tianyue) show nice power.

Challenge under overall null of no treatment effect

- Under the overall null, the oracle choice α_0 is not unique/defined.
- Therefore, $\hat{\alpha}(P_n)$ will stay random as sample size grows.
- We can still get an expansion for CV-TMLE but now in terms of $1/V \sum_v P_{n,v}^1 D_{\hat{\theta}_{\alpha_{n,v}}^*(\cdot), P_0}^*$, with a random index $\alpha_{n,v}$, causing lack of normality.
- Simulations show lack of normality but only slightly anti-conservative type-I error control.
- We can use a variance stabilized average over v as our test statistic (a la Alex Luedtke online stabilized one-step estimator!):

$$T_n \equiv n^{1/2} 1/V \sum_v \sigma_{\alpha_{n,v}, n}^{-1} \Psi_{\alpha_{n,v}}(P_{n,v}^*).$$

- To be continued!

THANK YOU!